2.4: Exact Differential Equations

Lets start with an example

$f(x,y) = x^2 \sin(y)$

Find a DE that will give some info about the equation (Partials!)

$2x\sin(y)\,dx + x^2\cos(y)\,dy = 0$

By using partial derivatives we obtain an Exact DE

$$\underbrace{2x\sin(y)\,dx}_{N(x,y)} + \underbrace{x^2\cos(y)\,dy}_{M(x,y)} = 0$$

$N_y = 2x\cos(y) \qquad M_x = 2x\cos(y)$

$\boxed{\text{same}}$

Criteria for exactness:

$N(x,y)\,dx + M(x,y)\,dy = 0$

If $\dfrac{\partial N(x,y)}{\partial y} = \dfrac{\partial M(x,y)}{\partial x}$

then the DE is exact

Consider: $3x^2\,dx + 2xy\,dy = 0$

Is it exact?

$\dfrac{\partial}{\partial y}(3x^2) = 0 \qquad \dfrac{\partial}{\partial x} = (2xy) = 2y$

not exact
$0 \neq 2y$

2.4: Continued

Ex) pg 69 #21 $(x+y)^2 dx + (2xy + x^2 - 1) dy = 0 \quad y(1)=1$

$M(x,y)\,dx + N(x,y)\,dy = 0$

Is it exact? yes

$\frac{d}{dy}(x+y)^2 = 2(x+y)$ ✓

$\frac{d}{dx}(2xy + x^2 - 1) = 2y + 2x$
$2(x+y)$ ✓

goal: we need to find a function $f(x,y)$ such that $\frac{d}{dx}(f(x,y)) = (x+y)^2$

and $\frac{d}{dy}(f(x,y)) = 2xy + x^2 - 1$

$\frac{d}{dx}(f(x,y)) = (x+y)^2$

$f(x,y) = \int (x+y)^2 dx$

$= \frac{(x+y)^3}{3} + \underbrace{g(y)}_{\text{instead of just a constant this could have involved } y}$

$\frac{d}{dy}\left(\frac{(x+y)^3}{3} + g(y)\right)$

$= \frac{3(x+y)^2}{3} + g'(y) = (x+y)^2 + g'(y)$

$$\frac{d}{dy}\left(\frac{(x+y)^3}{3} + g(y)\right)$$

$$= \frac{\cancel{3}(x+y)^2}{\cancel{3}} + g'(y)$$

$$= x^2 + 2xy + y^2 + g'(y)$$

match to $x^2 + 2xy - 1$

$$\cancel{x^2} + \cancel{2xy} + y^2 + g'(y) = \cancel{x^2} + \cancel{2xy} - 1$$

$$g'(y) = -1 - y^2$$

$$\int g'(y)\,dy = \int(-1 - y^2)\,dy$$

$$g(y) = -y - \frac{y^3}{3} + C$$

$$f(x,y) = \frac{(x+y)^3}{3} - y - \frac{y^3}{3} + C$$

$$= \frac{(1+1)^3}{3} - 1 - \frac{(1)^3}{3} + C = 0$$

$$\frac{8}{3} - 1 - \frac{1}{3} + C = 0$$

$$\frac{8-3-1}{3} + C = 0$$

$$\boxed{C = -\frac{4}{3}}$$

$$\boxed{f(x,y) = \frac{(x+y)^3}{3} - y - \frac{y^3}{3} + \left(-\frac{4}{3}\right)}$$

0) Standard form $N(x,y)dx + M(x,y)dy = 0$

Exact Equations: 1) verify they are exact (opposite partials)

2) Integrate either $N(x,y)dx$ or $M(x,y)dy$

I.E. $\int N(x,y)dx = f(x,y)$ missing $g(y)$

3) derivative $\frac{d}{dy}$ (since missing $g(y)$)

4) Match it to original (standard form)

5) Integrate

6) Find Initial condition

Ex) continued

$$(x+y)^2 dx + (2xy + x^2 - 1) dy = 0$$

$$N(x,y)dx + M(x,y)dy = 0$$

$$\int (2xy + x^2 - 1) dy = f(x,y)$$

$$f(x,y) = \frac{2xy^2}{2} + x^2y - y + g(x)$$

$$\frac{d}{dx}\left[xy^2 + x^2y - y + g(x)\right] = y^2 + 2xy + g'(x)$$

$$\cancel{y^2} + \cancel{2xy} + g'(x) = (x+y)^2$$

$$= x^2 + \cancel{2xy} + \cancel{y^2}$$

$$g'(x) = x^2$$

$$\int g'(x)dx = \int x^2 dx$$

$$g(x) = \frac{x^3}{3} + C$$

$$\boxed{f(x,y) = xy^2 + x^2y - y + \frac{x^3}{3} + C}$$

$$(1)(1)^2 + (1)^2(1) - 1 + \frac{1^3}{3} + C = 0 \qquad \frac{4}{3} + C = 0 \qquad \boxed{C = -\frac{4}{3}}$$

Ex) #20 $(x+2)^2 \frac{dy}{dx} = 5 - 8y - 4xy$

$(x+2)^2 dy = (5 - 8y - 4xy)\,dx$

$(5 - 8y - 4xy)\,dx - (x+2)^2 dy = 0$

$\frac{d}{dy}(5 - 8y - 4xy) = -8 - 4x$ $\qquad$ $\frac{d}{dx}(-(x+2)^2) = -2(x+2) = -2x - 4$

$\boxed{\text{Not Exact}}$

Ex) #14 $\left(1 - \frac{3}{y} + x\right)\frac{dy}{dx} + y = \frac{3}{x} - 1$

$\left(1 - \frac{3}{y} + x\right)dy + \left(y - \frac{3}{x} + 1\right)dx = 0$

$\frac{d}{dx}\left(1 - \frac{3}{y} + x\right) = 1$ $\qquad$ $\frac{d}{dy}\left(y - \frac{3}{x} + 1\right) = 1$

$\int \left(1 - \frac{3}{y} + x\right)dy = f(x,y)$

$f(x,y) = y - 3\ln|y| + xy + g(x)$

$\frac{d}{dx}\left(y - 3\ln|y| + xy + g(x)\right)$

$= y + g'(x)$

$\cancel{y} + g'(x) = \cancel{y} - \frac{3}{x} + 1$

$g'(x) = 1 - \frac{3}{x}$

$$\int y'(x)\,dx = \int \left(1-\frac{3}{x}\right)dx$$

$$y(x) = x - 3\ln|x| + C$$

$$\boxed{f(x,y) = y - 3\ln|y| + xy + x - 3\ln|x| + C}$$